

# On a problem of Terence Tao <sup>1</sup>

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**Abstract.** In this paper, we solve a problem of Terence Tao. We prove that for any  $K \geq 2$  and sufficiently large  $N$ , the number of primes  $p$  between  $N$  and  $(1 + \frac{1}{K})N$  such that  $|kp + ja^i + l|$  is composite for all  $1 \leq a, |j|, k \leq K$ ,  $1 \leq i \leq K \log N$  and  $l$  in any set  $L_N \subseteq \{-KN, \dots, KN\}$  of cardinality  $K$  with  $ja^i + l \neq 0$  is at least  $C_K \frac{N}{\log N}$ , where  $C_K > 0$  depending only on  $K$ .

Key Words: powers of  $a$ ; primes ; Selberg's sieve method.

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## 1. Introduction

Let  $p$  be a prime and  $n$  be a nonnegative integer. In 1934, Romanoff [12] proved that the set of positive odd integers which can be expressed in the form  $2^n + p$  has a positive proportion in the set of all positive odd numbers. In 1950, van der Corput [4] proved that there are a positive proportion odd integers not of the form  $2^n + p$ . In the same year, using covering congruences, Erdős [5] proved that there is an infinite arithmetic progression of positive odd integers each of which has no representation of the form  $2^n + p$ . In 1975, Cohen and Selfridge [3] proved that there exist infinitely many odd numbers which are neither the sum nor the difference of a power of two and a prime power.

Recently, using Selberg's sieve method, Tao [16] proved that for any  $K \geq 2$  and sufficiently large  $N$ , the number of primes  $p$  between  $N$  and  $(1 + \frac{1}{K})N$  such that  $|kp \pm ja^i|$  is composite for all  $1 \leq a, j, k \leq K$  and  $1 \leq i \leq K \log N$ , is at least  $C_K \frac{N}{\log N}$ , where  $C_K$  is a constant depending only on  $K$ .

On the other hand, Tao [16] posed the following problem:

For any  $K \geq 2$  and sufficiently large  $N$ , the number of primes  $p$  between  $N$  and  $(1 + \frac{1}{K})N$  such that  $|kp + ja^i + l|$  is composite for all  $1 \leq a, |j|, k \leq K$ ,  $1 \leq i \leq K \log N$  and  $l$  in some set  $L = L_N \subseteq \{-KN, \dots, KN\}$  of cardinality at most  $K$  is at least  $C_K \frac{N}{\log N}$ , where  $C_K$  is a constant depending only on  $K$ .

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Using Tao's idea, in this paper we shall solve the above Tao's problem. More precisely, we establish

**Theorem 1.** For any  $K \geq 2$  and sufficiently large  $N$ , the number of primes  $p$  between  $N$  and  $(1 + \frac{1}{K})N$  such that  $|kp + ja^i + l|$  is composite for all  $1 \leq a, |j|, k \leq K, 1 \leq i \leq K \log N$  and  $l$  in any set  $L_N \subseteq \{-KN, \dots, KN\}$  of cardinality  $K$  with  $ja^i + l \neq 0$  is at least  $C_K \frac{N}{\log N}$ , where  $C_K > 0$  depending only on  $K$ .

**Remark 1.** Let  $p(K)$  be a prime with  $p(K) > K$ . In Theorem 1, we can take  $L_N = \{p(K), \dots, Kp(K)\}$ . Moreover, we can take  $L_N = \{K! + 1, \dots, (2K - 1)! + 1\}$ .

**Remark 2.** From Theorem 1, we know that for any  $K \geq 2$  and sufficiently large  $N$ , the number of primes  $p$  between  $N$  and  $(1 + \frac{1}{K})N$  such that  $|kp + ja^i + l|$  is composite for all  $2 \leq k \leq K, 1 \leq a, |j| \leq K, 1 \leq i \leq K \log N$  and  $l$  in any set  $L_N \subseteq \{-KN, \dots, KN\}$  of cardinality  $K$  is at least  $C_K \frac{N}{\log N}$ , where  $C_K > 0$  depending only on  $K$ .

## 2. Proofs

In this paper,  $p, q, p_{i,j}, q_{i,j}$  are all primes, and the implied constants in  $\ll, \gg$  are all absolute.

**Lemma 1 [13].** Let  $x \geq 2$ . Then

$$\log \log x < \sum_{p \leq x} \frac{1}{p} < \log \log x + 1.$$

**Lemma 2 [13].** Let  $x \geq 59$ . Then

$$\frac{x}{\log x} \left(1 + \frac{1}{2 \log x}\right) < \pi(x) < \frac{x}{\log x} \left(1 + \frac{3}{2 \log x}\right).$$

By the Brun's theorem, we get

**Lemma 3.** There exists a constant  $A$  such that

$$\sum_{p, mp+1, \text{ primes}} \frac{1}{p} \leq A$$

for every  $1 < m \leq M$ , where the constant  $A$  depending only on  $M$ .

**Lemma 4.** For any  $2 \leq a \leq K$  and  $M > 12K^3$ , there exists a set  $P_a$  of some primes  $p_{a,t}$  in  $[\exp \exp((2^{a-1} - 1)(A + 1)M), \exp \exp((2^a - 1)(A + 1)M))$  with the following properties:

(1). For any  $p_{a,t}$ , there exists a prime  $q_{p_{a,t}}$  such that  $a^{p_{a,t}} \equiv 1 \pmod{q_{p_{a,t}}}$  and  $q_{p_{a,t}} \geq Mp_{a,t}$ .

(2). For each  $a$ , we have  $\sum_t \frac{1}{p_{a,t}} \in [M - 3, M]$ .

**Proof.** For  $a = 2$ , let  $P_2^*$  be the set of primes in the interval  $[\exp \exp((A + 1)M), \exp \exp(3(A + 1)M))$  satisfying that  $mp + 1$  is composite for every  $1 \leq m \leq M$ .

By Lemma 1 and Lemma 3, we have

$$\begin{aligned}
\sum_{p \in P_2^*} \frac{1}{p} &\geq \sum_{\exp \exp((A+1)M) \leq p < \exp \exp(3(A+1)M)} \frac{1}{p} \\
&\quad - \sum_{m=1}^{m=M} \sum_{\substack{\exp \exp((A+1)M) \leq p < \exp \exp(3(A+1)M), \\ mp+1, \text{ primes}}} \frac{1}{p} \\
&\geq 3(A+1)M - (A+1)M - 2 - MA \\
&\geq 2M - 2.
\end{aligned}$$

So, we can find a set  $P_2$  in  $P_2^*$  with  $\sum_t \frac{1}{p_{2,t}} \in [M-3, M]$ . Let  $q_{p_{2,t}}$  be the largest prime factor  $2^{p_{2,t}} - 1$ . We know all  $q_{p_{2,t}}$  are distinct. By the Fermat's little theorem, we know that  $p_{2,t}$  divides  $q_{p_{2,t}} - 1$ . On the other hand, we know that  $mp_{2,t} + 1$  is composite for every  $1 \leq m \leq M$ . Thus, we get  $q_{p_{2,t}} \geq Mp_{2,t}$ .

Now, suppose that  $a > 2$  and we have chosen disjoint finite sets of primes  $P_2, \dots, P_{a-1}$  with the stated properties.

Let  $P_a^*$  be the set of primes in the interval  $[\exp \exp((2^{a-1} - 1)(A+1)M), \exp \exp((2^a - 1)(A+1)M))$  satisfying that  $mp + 1$  is composite for every  $1 \leq m \leq M$ .

Let  $\omega_a = \prod_{2 \leq i < a} \prod_t (q_{p_{i,t}} - 1)$ . Note  $q_{p_{i,t}} | i^{p_{i,t}} - 1$ , by Lemma 2, we get

$$\begin{aligned}
&\frac{\log(\omega_a)}{\log a} \\
&\leq \sum_{2 \leq i < a} \sum_t \frac{\log(i^{p_{i,t}} - 1)}{\log a} \\
&\leq \sum_{2 \leq i < a} \sum_t p_{i,t} \frac{\log i}{\log a} \\
&\leq \sum_{2 \leq i < a} \sum_t p_{i,t} \\
&\leq \sum_{p \leq \exp \exp((2^{a-1} - 1)(A+1)M)} p \\
&\leq 2 \exp(-(2^{a-1} - 1)(A+1)M) \exp(2 \exp((2^{a-1} - 1)(A+1)M)).
\end{aligned}$$

Thus, we have  $\Omega(\omega_a) \leq \exp(2 \exp((2^{a-1} - 1)(A+1)M))$ .

Moreover, by Lemma 2, there exists at least  $\exp(2 \exp((2^{a-1} - 1)(A+1)M))$  primes in the interval  $[1, \exp(4 \exp((2^{a-1} - 1)(A+1)M))]$ .

So, we get

$$\sum_{p | \omega_a} \frac{1}{p} \leq \sum_{p \leq \exp(4 \exp((2^{a-1} - 1)(A+1)M))} \frac{1}{p} \leq (2^{a-1} - 1)(A+1)M + \log 4 + 1.$$

By Lemma 1 and Lemma 3, we have

$$\begin{aligned}
& \sum_{p \in P_a^*, p \nmid \omega_a} \frac{1}{p} \\
& \geq \sum_{\exp \exp((2^{a-1}-1)(A+1)M) \leq p < \exp \exp((2^a-1)(A+1)M)} \frac{1}{p} \\
& - \sum_{m=1}^{m=M} \sum_{\exp \exp((2^{a-1}-1)(A+1)M) \leq p < \exp \exp((2^a-1)(A+1)M), mp+1, \text{ primes}} \frac{1}{p} \\
& - \sum_{p \mid \omega_a} \frac{1}{p} \\
& \geq (2^a - 1)(A + 1)M - 1 - (2^{a-1} - 1)(A + 1)M - 1 - MA - (2^{a-1} - 1)(A + 1)M - \log 4 - 1 \\
& \geq M - 3.
\end{aligned}$$

Since  $p_{a,t} \mid q_{p_{a,t}} - 1$  and  $p_{a,t} \nmid \omega_a$ , we know all these  $q_{p_{a,t}}$  are distinct.

Similar to  $a = 2$ , we can choose a set  $P_a$  in  $P_a^*$  with the stated properties.

This completes the proof of Lemma 4.

**Proof of Theorem 1.** Let

$$R = \{(j, k, l) : 1 \leq |j|, k \leq K, l \in L_N\}$$

and take  $p$  and  $q_p$  as in Lemma 4.

By  $\sum_{p \in P_a} \frac{1}{p} \in [M - 3, M]$ , we may partition  $P_a = \bigcup_{j, k, l} P_{a, j, k, l}$  in such a way that

$$\sum_{p \in P_{a, j, k, l}} \frac{1}{p} \in [\frac{M}{4K^3}, \frac{M}{3K^3}].$$

Let  $W$  be the quantity  $W = \prod_p q_p$ . For  $p \in P_{a, j, k, l}$ , let  $I(a, j, k, l)$  be the smallest integer  $i \geq 0$  such that  $ja^i + l \not\equiv 0 \pmod{q_p}$ , we know that  $I(a, j, k, l) = 0, 1$ .

By the Chinese remainder theorem, we can take  $(b, W) = 1$  satisfying

$$kb + ja^{I(a, j, k, l)} + l \equiv 0 \pmod{q_p}$$

for every  $p \in P_{a, j, k, l}$ ,  $2 \leq a \leq K$ , and  $(j, k, l) \in R$ .

Let

$$\begin{aligned}
Q &= \#\{N \leq m \leq (1 + K^{-1})N : m \equiv b \pmod{W}, m \text{ prime, but } |km + ja^i + l| \text{ composite for all} \\
& 1 \leq i < K \log N, 1 \leq a \leq K, (j, k, l) \in R\}, \\
Q_N &= \#\{N \leq m \leq (1 + K^{-1})N : m \equiv b \pmod{W}, m \text{ prime}\},
\end{aligned}$$

and

$$Q_{N,a,i,j,k,l} = \#\{N \leq m \leq (1+K^{-1})N : m \equiv b \pmod{W}, m, |km + ja^i + l|, \text{primes}\}.$$

Similar to the proof of Theorem 1.2 [16], we get

$$Q \geq Q_N - \sum_{a=2}^K \sum_{1 \leq i < K \log N} \sum_{(j,k,l) \in R} Q_{N,a,i,j,k,l} - \sum_{(j,k,l) \in R} Q_{N,1,1,j,k,l} - O(\log N).$$

From the prime number theory in arithmetic progressions, we have

$$Q_N \geq c_1 \frac{N}{W \log N} \prod_{q|W} \left(1 - \frac{1}{q}\right).$$

Let  $P^* = \{p : K < p < N^{\frac{1}{8}}, (p, W) = 1\}$ .

By the Selberg's sieve method, we have

$$\begin{aligned} Q_{N,a,i,j,k,l} &= \#\{N \leq m \leq (1+K^{-1})N : m \equiv b \pmod{W}, m, |km + ja^i + l|, \text{primes}\} \\ &\leq \#\{N \leq m \leq (1+K^{-1})N : m \equiv b \pmod{W}, m > N^{\frac{1}{8}}, |km + ja^i + l| > N^{\frac{1}{8}}, \text{primes}\} + 2N^{\frac{1}{8}} \\ &\leq \#\{1 \leq r \leq (1+K^{-1})\frac{N}{W} + 1 : Wr + b > N^{\frac{1}{8}}, |kW r + kb + ja^i + l| > N^{\frac{1}{8}}, \text{primes}\} + 2N^{\frac{1}{8}} \\ &\leq \#\{1 \leq r \leq (1+K^{-1})\frac{N}{W} + 1 : ((Wr + b)(kW r + kb + ja^i + l), p) = 1, p \in P^*\} + 2N^{\frac{1}{8}} \\ &\ll \frac{N}{W} \prod_{p \nmid ja^i + l, p \in P^*} \left(1 - \frac{2}{p}\right) \prod_{p \mid ja^i + l, p \in P^*} \left(1 - \frac{1}{p}\right) \\ &\ll \frac{N}{W} \prod_{p \in P^*} \left(1 - \frac{2}{p}\right) \prod_{p \mid ja^i + l, p \in P^*} \left(1 - \frac{1}{p}\right) \left(1 - \frac{2}{p}\right)^{-1} \\ &\ll \frac{N}{W \log^2 N} \prod_{3 \leq p \leq K} \left(1 - \frac{2}{p}\right)^{-1} \prod_{K < q \mid W} \left(1 - \frac{2}{q}\right)^{-1} \prod_{K < p \mid ja^i + l, p \nmid W} \left(1 + \frac{1}{p}\right) \\ &\ll \frac{N}{W \log^2 N} \prod_{3 \leq p \leq K} \left(1 - \frac{2}{p}\right)^{-1} \prod_{K < q \mid W} \left(1 - \frac{2}{q}\right)^{-1} \prod_{K < p \mid ja^i + l} \left(1 + \frac{1}{p}\right). \end{aligned}$$

Now suppose that  $2 \leq a \leq K, (j, k, l) \in R$ .

Note that if  $i \equiv I(a, j, k, l) \pmod{p}$  for some  $p \in P_{a,j,k,l}$ , then  $q_p | km + ja^i + l$ , so  $q_p = |km + ja^i + l|$ .

Thus, we have

$$\sum_{1 \leq i \leq K \log N, i \equiv I(a, j, k, l) \pmod{p} \text{ for some } p \in P_{a,j,k,l}} Q_{N,a,i,j,k,l} \ll \log N$$

Let  $e_{a,j,l}(d)$  denote the smallest positive integer  $i$  such that  $ja^i + l \equiv 0 \pmod{d}$ . Since  $p|d \Rightarrow K < p$ , we know that  $d|ja^i + l$  if and only if  $e_{a,j,l}(d)|i$ .

Let

$$E(x) = \sum_{0 < k \leq x} \sum_{\mu^2(d)=1, e_{a,j,k,l}(d)=k, p|d \Rightarrow K < p} \frac{2^{\omega(d)}}{d}$$

Similar to the proof of Lemma 7.8 [7], we have

$$E(x) \ll \log^2 x.$$

By partial summation, we have

$$\sum_{0 < k \leq x} \frac{1}{k} \sum_{\mu^2(d)=1, e_{a,j,k,l}(d)=k, p|d \Rightarrow K < p} \frac{2^{\omega(d)}}{d} \ll 1.$$

So, we get

$$\sum_{\mu^2(d)=1, p|d \Rightarrow K < p} \frac{2^{\omega(d)}}{de_{a,j,l,k}(d)} = \sum_{0 < k} \frac{1}{k} \sum_{\mu^2(d)=1, e_{a,j,k,l}(d)=k, p|d \Rightarrow K < p} \frac{2^{\omega(d)}}{d} \ll 1.$$

By the Cauchy-Schwarz inequality and the Selberg's sieve method, we know

$$\begin{aligned} & \sum_{1 \leq i < K \log N, p \in P_{a,j,k,l} \Rightarrow p \nmid (i - I(a,j,k,l))} \prod_{K < p | ja^i + l} \left(1 + \frac{1}{p}\right) \\ & \ll \left( \sum_{1 \leq i < K \log N, p \in P_{a,j,k,l} \Rightarrow p \nmid (i - I(a,j,k,l))} 1 \right)^{\frac{1}{2}} \left( \sum_{1 \leq i < K \log N} \prod_{K < p | ja^i + l} \left(1 + \frac{1}{p}\right)^2 \right)^{\frac{1}{2}} \\ & \ll (K \log N \prod_{p \in P_{a,j,k,l}} \left(1 - \frac{1}{p}\right))^{\frac{1}{2}} \left( \sum_{1 \leq i < K \log N} \prod_{K < p | ja^i + l} \frac{2}{p} \right)^{\frac{1}{2}} \\ & \ll (K \log N \prod_{p \in P_{a,j,k,l}} \left(1 - \frac{1}{p}\right))^{\frac{1}{2}} \left( \sum_{1 \leq i < K \log N} \prod_{\mu^2(d)=1, d|ja^i + l, p|d \Rightarrow K < p} \frac{2^{\omega(d)}}{d} \right)^{\frac{1}{2}} \\ & \ll (K \log N \prod_{p \in P_{a,j,k,l}} \left(1 - \frac{1}{p}\right))^{\frac{1}{2}} \left( \sum_{\mu^2(d)=1, p|d \Rightarrow K < p} \sum_{1 \leq i < K \log N, d|ja^i + l} \frac{2^{\omega(d)}}{d} \right)^{\frac{1}{2}} \\ & \ll (K \log N \prod_{p \in P_{a,j,k,l}} \left(1 - \frac{1}{p}\right))^{\frac{1}{2}} \left( \sum_{\mu^2(d)=1, p|d \Rightarrow K < p} \frac{(K \log N) 2^{\omega(d)}}{de_{a,j,k,l}(d)} \right)^{\frac{1}{2}} \\ & \ll K \log N \prod_{p \in P_{a,j,k,l}} \left(1 - \frac{1}{p}\right)^{\frac{1}{2}} \end{aligned}$$

So, we have

$$\begin{aligned} & \sum_{\substack{1 \leq i \leq K \log N, i \not\equiv I(a,j,k,l) \pmod{p} \text{ for any } p \in P_{a,j,k,l}}} Q_{N,a,i,j,k,l} \\ & \ll \frac{KN}{W \log N} \prod_{3 \leq p \leq K} \left(1 - \frac{2}{p}\right)^{-1} \prod_{q|W} \left(1 - \frac{2}{q}\right)^{-1} \prod_{p \in P_{a,j,k,l}} \left(1 - \frac{1}{p}\right)^{\frac{1}{2}}. \end{aligned}$$

Thus, we get

$$\begin{aligned} & \sum_{1 \leq i \leq K \log N} Q_{N,a,i,j,k,l} \\ & \ll \frac{KN}{W \log N} \prod_{3 \leq p \leq K} \left(1 - \frac{2}{p}\right)^{-1} \prod_{q|W} \left(1 - \frac{2}{q}\right)^{-1} \prod_{p \in P_{a,j,k,l}} \left(1 - \frac{1}{p}\right)^{\frac{1}{2}} + \log N. \end{aligned}$$

By Lemma 1, we get

$$\begin{aligned} Q & \geq c_1 \frac{N}{W \log N} \prod_{q|W} \left(1 - \frac{1}{q}\right)^{-1} - c_2 \frac{KN}{W \log N} \prod_{3 \leq p \leq K} \left(1 - \frac{2}{p}\right)^{-1} \prod_{q|W} \left(1 - \frac{2}{q}\right)^{-1} \sum_{a=2}^K \sum_{(j,k,l) \in R} \prod_{p \in P_{a,j,k,l}} \left(1 - \frac{1}{p}\right)^{\frac{1}{2}} \\ & - \sum_{(j,k,l) \in R} Q_{N,1,0,j,k,l} - O(\log N) \\ & \geq c_1 \frac{N}{W \log N} \prod_{q|W} \left(1 - \frac{1}{q}\right)^{-1} - c_3 \frac{KN}{W \log N} \prod_{3 \leq p \leq K} \left(1 - \frac{2}{p}\right)^{-1} \prod_{q|W} \left(1 - \frac{1}{q}\right)^{-2} \sum_{a=2}^K \sum_{(j,k,l) \in R} \prod_{p \in P_{a,j,k,l}} \left(1 - \frac{1}{p}\right)^{\frac{1}{2}} \\ & \geq c_1 \frac{N}{W \log N} \prod_{q|W} \left(1 - \frac{1}{q}\right)^{-1} (1 - c_4 \prod_{3 \leq p \leq K} \left(1 - \frac{2}{p}\right)^{-1} \prod_{q|W} \left(1 - \frac{1}{q}\right)^{-1} \sum_{a=2}^K \sum_{(j,k,l) \in R} \prod_{p \in P_{a,j,k,l}} \left(1 - \frac{1}{p}\right)^{\frac{1}{2}}) \\ & \geq c_1 \frac{N}{W \log N} \prod_{q|W} \left(1 - \frac{1}{q}\right)^{-1} (1 - c_5 (\log K)^2 \sum_{a=2}^K \sum_{(j,k,l) \in R} \exp(\sum_{q|W} \frac{1}{q} - \sum_{p \in P_{a,j,k,l}} \frac{1}{2p})) \\ & \geq c_1 \frac{N}{W \log N} \prod_{q|W} \left(1 - \frac{1}{q}\right)^{-1} (1 - 2c_5 (\log K)^2 K^4 \exp(K - \frac{M}{8K^3})). \end{aligned}$$

Taking  $M > \max\{12K^3, 8K^4 + 8K^3 \log(4c_5 (\log K)^2 K^4)\}$ , we get  $Q \geq C \frac{N}{W \log N} \prod_{q|W} (1 - \frac{1}{q})^{-1}$ , where the constant  $C$  is absolute.

This completes the proof of the Theorem 1.

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